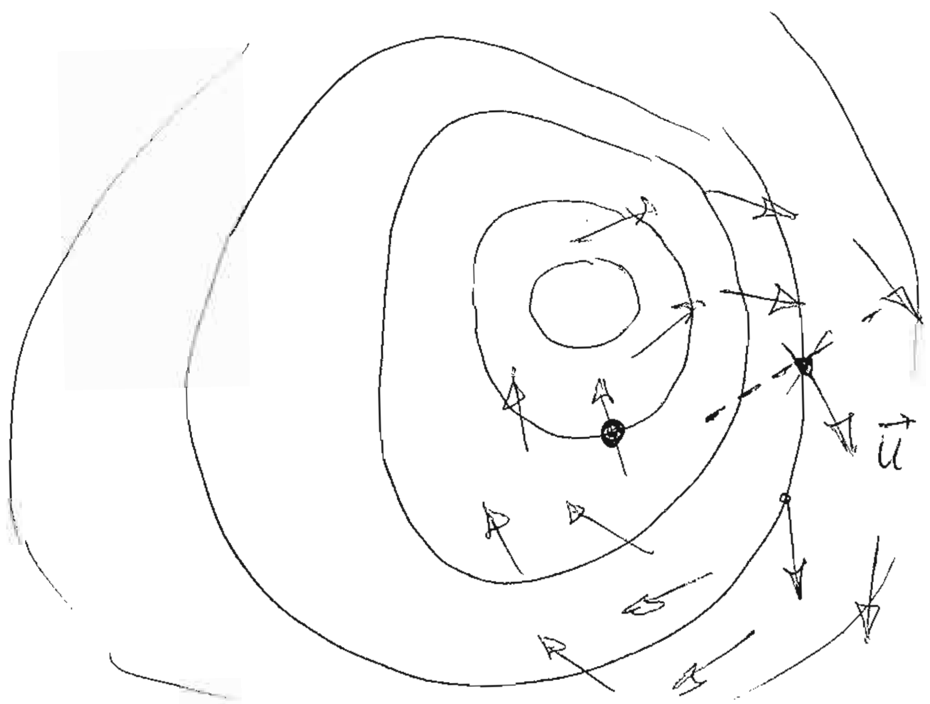
# Non-holonomic constraints

Here's a new type of optimization problem:

find extremal points of $f(x,y)$ such that $x,y$ may only vary $\perp$ to the vector field $\vec{u}(x,y)$:




equivalent: find $x,y$ such that

$$\vec{\nabla} f \parallel \vec{u} \Rightarrow \vec{\nabla} f = \lambda \vec{u} \text{ for some scalar } \lambda$$

If we could construct a function $g(x,y)$ such that $\vec{\nabla} g = \vec{u}$, then this type of problem would be equivalent to what we studied previously. However, it is often the case that this is not possible (e.g. when $\vec{u}$ corresponds to a non-conservative force). On the other hand, the device of Lagrange multipliers works exactly as before!

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With more independent variables $(x, y, z, \ldots)$ and several constraints we can use the same approach:

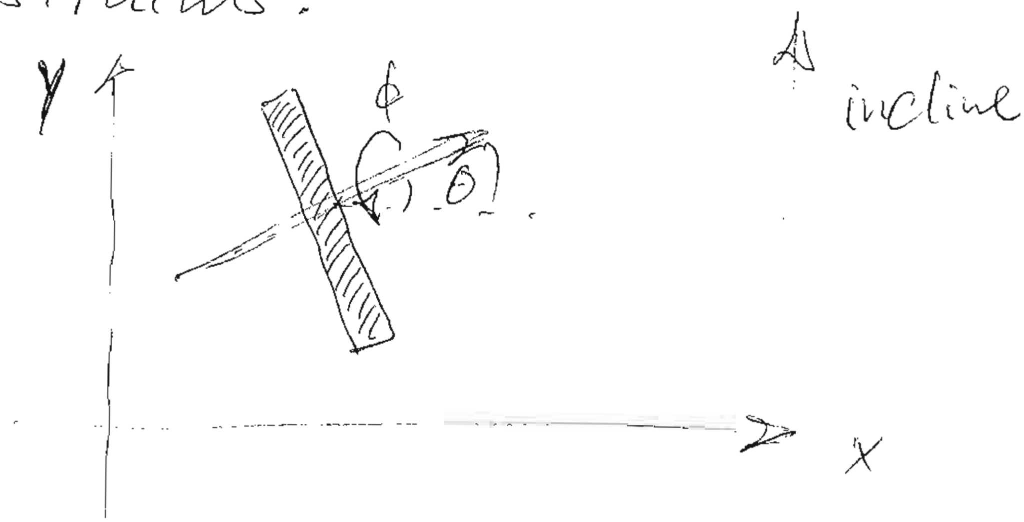
find a point $x, y, z, \ldots$ such that $f(x, y, z, \ldots)$ is extremal when the variations $\delta x, \delta y, \ldots$ are perpendicular to $\vec{u}, \vec{v}, \ldots$ is equivalent to, find $x, y, z, \ldots$ such that

$$\vec{\nabla} f(x,y,z,\ldots) = \lambda_1 \vec{u}(x,y,\ldots) + \lambda_2 \vec{v}(x,y,\ldots) + \cdots$$

for some scalars $\lambda_1, \lambda_2, \ldots$

In the "holonomic" case the vector fields $\vec{u}, \vec{v}, \ldots$ can be expressed as gradients of functions $g, h, \ldots$. When this is not possible we say the constraints are "non-holonomic".

Let's return the problem of the wheel that rolls without slipping on an inclined plane, which we said was a case of non-holonomic constraints.




Here is the rolling-without-slipping constraint:

$$\dot{x} = r\dot{\phi}\sin\theta \quad (1)$$

$$\dot{y} = -r\dot{\phi}\cos\theta \quad (2)$$

(1') $\quad \delta x(t) - r\sin\theta(t)\,\delta\phi(t) = 0$

(2') $\quad \delta y(t) + r\cos\theta(t)\,\delta\phi(t) = 0$

Let's focus on (2'). This tells us the variations are restricted to a linear space of one lower dimension.

Geometrically, the variations must be perpendicular to a particular constraint vector, analogous to $\vec{u}$ in the previous discussion.

Components of constraint vector:

$$\text{“}\vec{C}(t)\text{”} \begin{cases} 1 & \text{in component } \delta y(t) \\ r\cos\theta(t) & \text{in component } \delta\phi(t) \\ 0 & \text{all other components} \end{cases}$$

We will associate the Lagrange multiplier $\lambda(t)$ with this constraint.

Here is the Lagrangian of our wheel, expressed in terms of $\phi$, $\theta$, $y$ (one more than the number of degrees of freedom):

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_\theta \dot{\theta}^2 + \frac{1}{2} I_\phi \dot{\phi}^2 - mg \sin\alpha \, y$$

$$= \frac{1}{2}(mr^2 + I_\phi)\dot{\phi}^2 + \frac{1}{2} I_\theta \dot{\theta}^2 - mg \sin\alpha \, y$$

$$S[y(t), \phi(t), \theta(t)] = \int L \, dt$$

Let's evaluate the variational derivative ("gradient") of $S$, neglecting the constraint (so $y$, $\phi$, $\theta$, are treated as independent):

$$\frac{\delta S}{\delta y(t)} = -mg \sin\alpha$$

$$\frac{\delta S}{\delta \phi(t)} = -\frac{d}{dt}\left((mr^2 + I_\phi)\dot{\phi}\right)$$

$$\frac{\delta S}{\delta \theta(t)} = -\frac{d}{dt}\left(I_\theta \dot{\theta}\right)$$

We now impose the "gradient" condition

$$\vec{\nabla} S = \sum_t \lambda(t) \vec{U}(t)$$

The vector components in this equation that apply to a particular time $t$ just involve the single Lagrange multiplier $\lambda(t)$:

$\delta y(t)$ component:

$$-mg \sin\alpha = \lambda(t) \cdot 1 \qquad (A)$$

$\delta\phi(t)$ component:

$$-(mr^2 + I_\phi)\ddot{\phi} = \lambda(t)\, r \cos\theta(t) \qquad (B)$$

$\delta\theta(t)$ component:

$$-I_\theta \ddot{\theta} = \lambda(t) \cdot 0 \qquad (C)$$

These differential equations are easily solved:

$$(A) \Rightarrow \lambda = -mg\sin\alpha \quad \text{(constant in time)}$$

$$(C) \Rightarrow \theta(t) = \omega t + \theta_0$$

$$(B) \Rightarrow (mr^2 + I_\phi)\ddot{\phi} = mgr\sin\alpha\cos(\omega t + \theta_0)$$

(B) and (C) are the same equations we found previously, using Newton's laws directly.